Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-03-04

Conceptual questions

Question 1. Is it possible for different level curves of a function $f(x, y)$ to intersect?

Question 2. How are the direction and magnitude of the gradient vector related to level curves?

Question 3. If $\mathbf{r}(t)$ is a curve contained in the surface $f(x, y, z) = 0$, what is the angle between $\mathbf{r}'(t)$ and $\nabla f(\mathbf{r}(t))$?

Question 4. Fix a function $f(x, y)$, a number c, and a point (a, b) where $\nabla f(a, b) \neq \mathbf{0}$. How many unit vectors $\mathbf{u}$ are such that $D_{\mathbf{u}} f(a, b) = c$? Hint: The answer depends on $|c|$.

Computations

Problem 1. Find all points on the surface

$$x^2 + y^2 + 4z^2 - 2xz - 2yz - 2x + 2z = 1$$

which have a horizontal tangent plane.

Problem 2. Consider the equation $yz + x \ln y + z^3 = 0$.

(a) If $(x, y) = (3, 1)$, what value of z satisfies the equation?

(b) Near the point $(3, 1, a)$ (where a is your answer to the preceding part), the equation implicitly defines z as a function of x and y: $z = f(x, y)$. Compute $\nabla f(3, 1)$.

Problem 3.

(a) Suppose that $\mathbf{r}(t) = (x(t), y(t))$ is parametrized by arclength (recall that this means $|\mathbf{r}'(t)| = 1$; the particle is "moving at speed 1"). Show that the directional derivative of f in the direction of $\mathbf{r}'(t)$ is equal to $\frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{r}(t)))$. Hint: Use the chain rule.

(b) Consider the function

$$f(x, y) = \cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right).$$

Using the preceding part, compute $f_y(1, 0)$. Hint: Use the unit circle.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. No, at a point (a, b) it is not possible for $f(a, b)$ to equal two different values simultaneously.

Question 2. At a point (a, b), the gradient vector $\nabla f(a, b)$ is perpendicular to the level set of f passing through (a, b). The magnitude of the gradient vector is inversely proportional to the spacing between level sets.

Question 3. $\pi/2$; they are orthogonal.

Question 4. If $|c| > |\nabla f(a, b)|$ then none. If $|c| = |\nabla f(a, b)|$ then one. If $|c| < |\nabla f(a, b)|$ then two.

Answers to computations

Problem 1. The tangent plane is horizontal when the gradient has the form $\langle 0, 0, ?\rangle$. That is, we should find the points on the surface where $F_x = F_y = 0$. This system of equations results in the solutions $(x, y, z) = (2, 1, 1), (0, -1, -1)$.

Problem 2.

(a) $z = 0$

(b) Use the Implicit Function Theorem to compute $\partial z/\partial x$ and $\partial z/\partial y$, which are the components of ∇f. The final answer is $\langle 0, -3\rangle$.

Problem 3.

(a) The directional derivative in the direction of $\mathbf{r}'(t)$ is

$$\begin{aligned} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)) \end{aligned}$$

where in the last step the chain rule was used.

(b) Use $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, which is parametrized by arclength and has $\mathbf{r}'(0) = \langle 0, 1\rangle$. So by the preceding part,

$$\begin{aligned} f_y(1, 0) &= D_{\langle 0,1\rangle} f(x, y) \\ &= \left.\left(\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\right)\right|_{t=0} \\ &= \left.\left(\frac{\mathrm{d}}{\mathrm{d}t}(2t)\right)\right|_{t=0} \\ &= 2. \end{aligned}$$